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National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Purchasing Office
Huntsville, Alabama

Attention: PR-EC

Re : Contract NAS8-20175
Subject: Report for period from 1 April, 1968 to 1 April, 1969.

Gentlemen:

I. Introduction

This report summarizes the essential results of our studies during the period April 1, 1968 to April 1, 1969, aimed at applying the perturbation methods of celestial mechanics to the rigid body problem with particular emphasis on the problem of the motion of an artificial Earth satellite about its center of mass. During this period, we were able to express the Hamiltonian for the triaxial body in terms of variables in which it is readily separable. This, in turn, permitted the introduction of a canonical transformation to new parameters which are constants in the torque-free motion. The equations of transformation were then inverted to enable us to express the original Euler angles and associated conjugate momenta in terms of the canonical constants and the time. Thus, we were able to set up the formalism for studying perturbations of a triaxial rigid body within the Hamilton-Jacobi framework. The essential details follow.

II. Rectangular Coordinate Systems and Euler Angles

Let O represent the center of mass of the rigid body. Choose a space-fixed rectangular system $O\xi\eta\zeta$ such that the positive ζ-axis lies along the angular momentum vector $\underline{H}$ and in the sense of $\underline{H}$. Consider a plane through the center of mass and perpendicular to the ζ-axis. This plane intersects the fundamental plane of the space-fixed, but otherwise arbitrary, rectangular frame $Ox^*y^*z^*$ in a line of nodes ON, as shown in Figure 1. The ξ-axis is chosen to lie along the line of nodes, its positive sense being arbitrarily chosen. Then the η-axis is chosen to form a right-handed system.

Let $Ox'y'z'$ be a body-fixed (principal axes) rectangular frame and let ϕ^*, θ', and ϕ' represent the Euler angles relating the $Ox'y'z'$ and $O\xi\eta\zeta$ systems. We will refer to the $x'y'$-plane as the body-fixed plane. The angle ψ^* is the angle between the x^* and ξ-axes, measured in the x^*y^*-plane while the angle θ^* is the angle between the positive z^* and ζ-axes.

III. The solution of the Hamilton-Jacobi equation associated with a triaxial body problem with no external forces

(A) Hamilton function and canonical equations

Although the eventual goal is to give a complete discription of the motion in the $Ox^*y^*z^*$ system, the description of the motion will first be given in the $O\xi\eta\zeta$-system. In this manner, a straightforward, coherent approach to the problem and its solution, can be presented.

Let

$$= \begin{pmatrix} p_{\psi'} \\ p_\theta \\ p_\phi \end{pmatrix} \qquad (a)$$

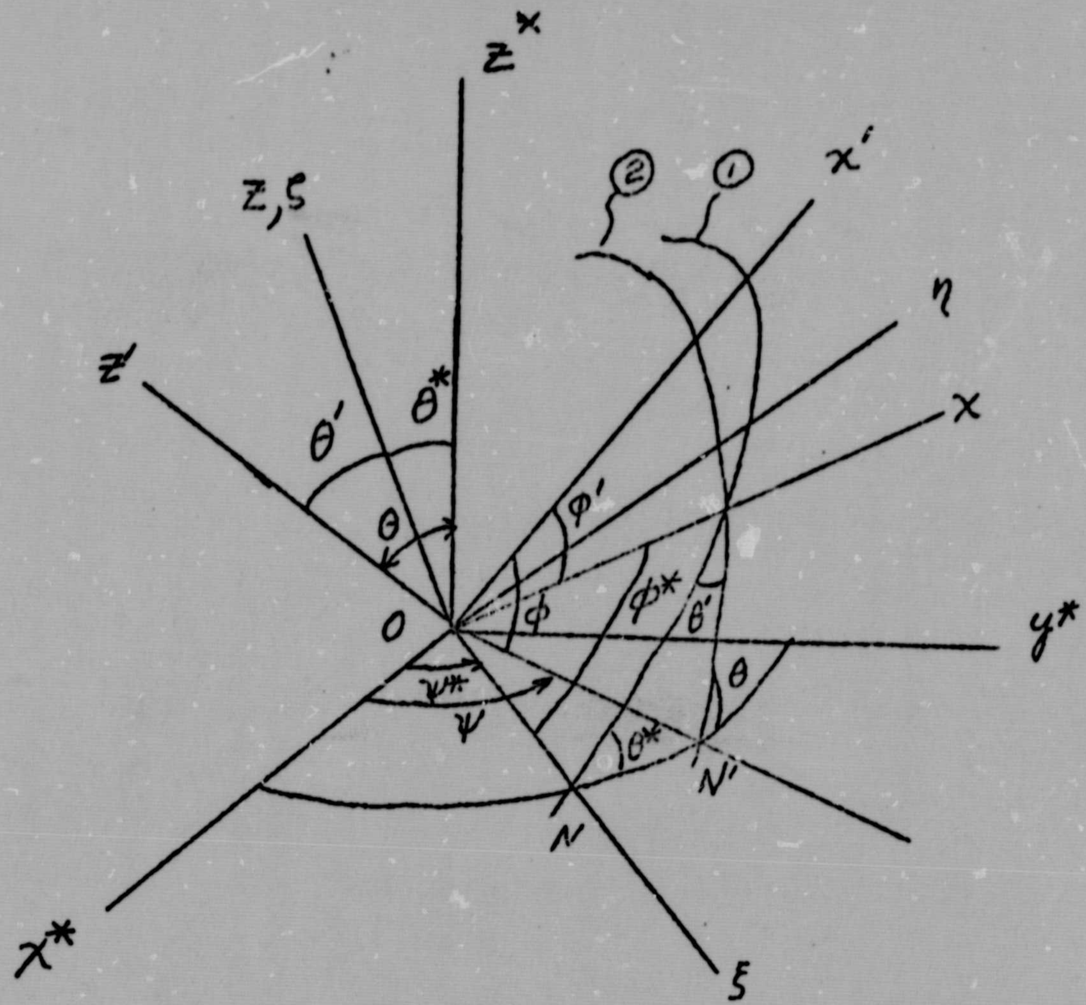

Figure 1

1 angular momentum plane

2 body-fixed plane

$Ox^*y^*z^*$ space-fixed axes

$Ox'y'z'$ body-fixed axes

$Oxyz$ angular momentum axes

$$\xi = \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}, \qquad \text{(b)}$$

$$H^* = \begin{pmatrix} h_x^* \\ h_y^* \\ h_z^* \end{pmatrix}, \qquad \text{(c)}$$

$$H' = \begin{pmatrix} h_x' \\ h_y' \\ h_z' \end{pmatrix}, \qquad \text{(d)} \qquad (1)$$

where $\underline{P}$ represents the conjugate momenta matrix while $\underline{H}$ and $\underline{H}'$ represent the angular momentum w.r.t. space-fixed and body fixed axes, respectively. All matrices involved are to be found explicitly in our reports for September, October, and November, 1965. A recapitulation of formulas is given below to help the reader to follow the subsequent discussion.

We have

$$P = X\xi \ , \qquad (2)$$

or explicitly

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} (A \sin^2\phi + B \cos^2\phi) \sin^2\theta + C \cos^2\theta, & (A-B) \sin\phi \cos\phi \sin\theta, & C \cos\phi \\ (A-B) \sin\phi \cos\phi \sin\theta, & A \cos^2\phi + B \sin^2\phi & 0 \\ C \cos \theta & 0 , & C \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} .$$

$$\underline{H}^* = T\,H' , \tag{3}$$

or

$$\begin{pmatrix} h_x^* \\ h_y^* \\ h_z^* \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\phi-\sin\psi\sin\phi\cos\theta, & -\cos\psi\sin\phi-\sin\psi\cos\phi\cos\theta, & \sin\psi\sin\theta \\ \sin\psi\cos\phi+\cos\psi\sin\phi\cos\theta, & -\sin\psi\sin\phi+\cos\psi\cos\phi\cos\theta, & -\cos\psi\sin\theta \\ \sin\phi\sin\theta, & \cos\phi\sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} h_x' \\ h_y' \\ h_z' \end{pmatrix}$$

$$P = N^T \underline{H}' , \tag{4}$$

or.

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_x' \\ h_y' \\ h_z' \end{pmatrix}$$

$$P = M^T \underline{H}^* , \tag{5}$$

or

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\psi & \sin\psi & 0 \\ \sin\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_x^* \\ h_y^* \\ h_z^* \end{pmatrix} .$$

and from (3), we have

$$\underline{H}' = T^I \underline{H}^* , \tag{6}$$

or explicitly

$$\begin{pmatrix} h_{x'} \\ \\ h_{y'} \\ \\ h_{z'} \end{pmatrix} \begin{aligned} &= p_\theta \cos\phi + \frac{\sin\phi}{\sin\theta}(p_\psi - p_\phi \cos\theta) , \\ &= -p_\theta \sin\phi + \frac{\cos\phi}{\sin\theta}(p_\psi - p_\phi \cos\theta) , \\ &= p_\phi = h \cos\theta' . \end{aligned}$$

In the $O\xi\eta\zeta$-system, the angular momentum can be written as

$$\underline{H} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} . \tag{7}$$

Using (5), with ψ, θ, ϕ replaced by ϕ^*, θ', ϕ', respectively, we obtain

$$\begin{pmatrix} p_{\phi^*} \\ p_{\theta'} \\ p_{\phi'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi^* & \sin\phi^* & 0 \\ \sin\phi^* \sin\theta' & -\cos\phi^* \sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{8}$$

in

$$p_{\phi^*} = h ,$$

$$p_{\theta'} = 0 ,$$

$$p_{\phi'} = h \cos\theta' .$$

Similarly, after using (8), from (6), with ψ', θ, ϕ replaced by ϕ^*, θ', ϕ', respectively, we obtain

$$\left(\begin{aligned} h_{x'} &= \frac{\sin\phi'}{\sin\theta'}\,(p_{\phi^*} - p_{\phi'}\cos\theta') \,, \\ h_{y'} &= \frac{\cos\phi'}{\sin\theta'}\,(p_{\phi^*} - p_{\phi'}\cos\theta') \,, \\ h_{z'} &= p_{\phi'} \,. \end{aligned}\right) \tag{9}$$

Using (9), coupled with (8), we can write the kinetic energy (the Hamilton function) of the rigid body in the form

$$T = \frac{1}{2A} h_{x'}^2 + \frac{1}{2B} h_{y'}^2 + \frac{1}{2C} h_{z'}^2 = H \,,$$

or

$$H = T = \left(\frac{\sin^2\phi'}{2A} + \frac{\cos^2\phi'}{2B} \right) (p_{\phi^*}^2 - p_{\phi'}^2) + \frac{p_{\phi'}^2}{2C} \,, \tag{10}$$

where A, B and C are the moments of inertia of the rigid body referenced to the principal axes, $Ox'y'z'$. We assume that

$$A > B > C \,.$$

The associated canonical equations are

$$\dot{\phi}^* = \frac{\partial H}{\partial p_{\phi^*}} = h\left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right) \,, \qquad \text{(a)}$$

$$\dot{\phi}' = \frac{\partial H}{\partial p_{\phi'}} = -h\cos\theta' \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right) + \frac{p_{\phi'}}{C} \,, \qquad \text{(b)}$$

$$\dot{p}_{\phi^*} = -\frac{\partial H}{\partial \phi^*} = 0 \,, \qquad \text{(c)} \tag{11}$$

$$\dot{p}_{\phi'} = -\frac{\partial H}{\partial \phi'} = h^2\left(\frac{1}{B} - \frac{1}{A}\right) \sin\phi' \cos\phi' \sin^2\theta' = -h \sin\theta' \,\dot{\theta}', \quad \text{(d)}$$

$$p_{\theta}' = 0 , \quad \text{(e)}$$

$$\cos\theta' = \frac{p_{\phi}'}{h} . \quad \text{(f)}$$

(B) Description of the Motion in the $Ox^*y^*z^*$—system

In this section a set of relationships is given from which the description of the motion in the space fixed system $(\psi, \theta, \phi, p_{\psi}, p_{\theta}, p_{\phi})$ can be obtained completely from the description of the motion in the body-fixed system $(\phi^*, \theta', \phi', p_{\phi^*}, p_{\theta}', p_{\phi}')$.

From the elementary spherical trigonometry, we have

$$\left.\begin{aligned} \cos\theta &= \cos\theta' \cos\theta^* - \sin\theta' \sin\theta^* \cos\phi^* , \\ \sin\theta &= \sqrt{1 - \cos^2\theta} , \end{aligned}\right\} \quad \text{(a)}$$

$$\left.\begin{aligned} \sin\theta^* \sin\theta \cos(\psi - \psi^*) &= \cos\theta' - \cos\theta^* \cos\theta , \\ \sin\theta \sin(\psi - \psi^*) &= \sin\phi^* \sin\theta' , \end{aligned}\right\} \quad \text{(b)} \qquad (12)$$

$$\left.\begin{aligned} \sin\theta' \cos\theta' \cos(\phi - \phi') &= \cos\theta^* - \cos\theta' \cos\theta , \\ \sin\theta \sin(\phi - \phi') &= \sin\phi^* \sin\theta^* . \end{aligned}\right\} \quad \text{(c)}$$

Through the use of equations (3), (4) and (5), we can relate the variables p_{ψ} , p_{θ} , and p_{ϕ} to the variables p_{ϕ}' , p_{ϕ^*} , θ^* , ψ^* , and ϕ^* . Explicitly, we can write these relationships as follows:

$$p_{\theta} = -p_{\phi^*} \sin\theta' \sin(\phi - \phi') , \quad \text{(a)}$$

$$p_\psi = h\cos\theta^* , \qquad \text{(b)} \qquad (13)$$

$$p_\phi = p_{\phi'} . \qquad \text{(c)}$$

Since θ^* and ψ^* are prescribed constants, independent of each other and independent of ϕ^*, ϕ', p_{ϕ^*}, $p_{\phi'}$, the independent quantities $(\phi^*, \phi', \theta^*, \psi^*, p_{\phi^*}, p_{\phi'})$ serve to describe the motion of the triaxial body in the $OX^*Y^*Z^*$ system.

(C) Generator and equations of transformation[1]

The Hamilton-Jacobi equation associated with equation (10) is

$$\frac{1}{2}\left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)\left[\left(\frac{\partial S}{\partial \phi^*}\right)^2 - \left(\frac{\partial S}{\partial \phi'}\right)^2\right] + \frac{1}{2C}\left(\frac{\partial S}{\partial \phi'}\right)^2 + \frac{\partial S}{\partial t} = 0, \qquad (14)$$

from which the generator S of a canonical transformation is to be determined. A complete integral S of (14) can be obtained by separation of variables. We find that

$$S = -\alpha_1 t + h\,\phi^* + \alpha_3\,\psi^* + S_1(\phi') , \qquad (15)$$

where

$$\alpha_1 = H ,$$

$$h = p_{\phi^*} = \frac{\partial S}{\partial \phi^*} , \qquad (16)$$

$$\alpha_3 = p_{\psi^*} = \frac{\partial S}{\partial \psi^*} ,$$

are independent canonical variables. The function $S_1(\phi')$ is related to α_1 and h through the expression

$$S_1(\phi') = \int_{\phi_0'}^{\phi'} p_{\phi'}\, d\phi' , \qquad (17)$$

1 The variables $(\phi^*_*, \phi', {}^*, p_{\phi^*}, p_{\phi'}, p\ *)$ in which we replace θ^* by $\cos^{-1}\left(\frac{p_*}{h}\right)$ are introduced here (see VII for justification).

where

$$p_{\phi'} = \pm \sqrt{C\left(\frac{a' + b' \sin^2 \phi'}{c' + d' \sin^2 \phi'}\right)}\,, \tag{18}$$

and

$$\begin{aligned} a' &= A(2B\alpha_1 - h^2)\,, \\ b' &= h^2(A - B)\,, \\ c' &= A(B - C)\,, \\ d' &= C(A - B)\,. \end{aligned} \tag{19}$$

The complete set of equations of transformation from $(\psi^*, \phi^*, \phi', p_{\psi^*}, p_{\phi^*}, p_{\phi'})$ to $(\alpha_1, h, \alpha_3, \beta_1, \beta_2, \beta_3)$ are obtained from equation (15). They are

$$\beta_1 = -\frac{\partial S}{\partial \alpha_1} = t - L(\phi')\,. \tag{a}$$

$$\beta_2 = -\frac{\partial S}{\partial h} = M(\phi') - \phi^*, \tag{b}$$

$$\beta_3 = -\frac{\partial S}{\partial \alpha_3} = -\psi^*\,, \tag{c}$$

$$p_{\psi}^* = \frac{\partial S}{\partial \psi^*} = \alpha_3\,, \tag{d}$$

$$p_{\phi}^* = \frac{\partial S}{\partial \phi^*} = h\,, \tag{e}$$

$$p_{\phi'} = \frac{\partial S}{\partial \phi'} = \pm \sqrt{C\left(\frac{a' + b' \sin^2 \phi'}{c' + d' \sin^2 \phi'}\right)}\,, \tag{f}$$

where

$$L(\phi') = \pm AB\sqrt{C}\, I_2(\phi') ,$$
$$M(\phi') = \mp \sqrt{C}\, \alpha_2\, I_3(\phi') , \tag{21}$$

and

$$I_2(\phi') = \int_{\phi_0'}^{\phi'} \frac{d\phi'}{\sqrt{(a + b \sin^2\phi')(c + d \sin^2\phi')}} , \quad (a)$$
$$I_3(\phi') = \int_{\phi_0'}^{\phi'} \frac{[(A-B)\sin^2\phi' - A]\, d\phi'}{\sqrt{(a + b \sin^2\phi')(c + d \sin^2\phi')}} . \quad (b) \tag{22}$$

In three of the six of equations (20), the right-hand sides are preceded by $\pm$ symbols. The choice of the sign in these equations is determined by the choice of sign for $p_{\phi'}$. We have

$$P = M^T H , \tag{23}$$

or

$$\begin{pmatrix} p_{\phi^*} \\ p_{\theta'} \\ p_{\phi'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi^* & \sin\phi^* & 0 \\ \sin\phi^* \sin\theta' & -\cos\phi^* \sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} . \tag{23}$$

hence

$$p_{\phi'} = h \cos\theta' .$$

Thus $p_{\phi'}$ is taken positive or negative according as $\cos\theta'$ is positive or negative. We will assume that $0 < \theta' < \pi/2$. Thus equations (20) and (21) become

$$\left.\begin{aligned} & t - \beta_1 = L(\phi') \,, && \text{(a)} \\ & \phi^* + \beta_2 = M(\phi') \,, && \text{(b)} \\ & \beta_3 = -\psi^* \,, && \text{(c)} \\ & p_{\phi}' = \sqrt{C\left(\frac{a' + b' \sin^2\phi'}{c' + d' \sin^2\phi'}\right)} \,, && \text{(d)} \\ & p_{\phi}^* = h \,, && \text{(e)} \\ & p_{\psi}^* = \alpha_3 \,, && \text{(f)} \end{aligned}\right\} \quad (24)$$

where

$$\left.\begin{aligned} & L(\phi') = AB\sqrt{C}\;\; I_2(\phi') \,, && \text{(a)} \\ & M(\phi') = -h\sqrt{C}\;\; I_3(\phi') \,. && \text{(b)} \end{aligned}\right\} \quad (25)$$

IV. Inversion of the solution for the triaxial rigid body problem with no external forces

We have to invert (24) to express the variables $(\phi^*, \phi', \psi^*, p_{\phi}^*, p_{\phi}', p_{\psi}^*)$ in terms of the canonical constants α_1 , h , α_3 ,β_1,β_2,β_3 and time t . The inversion is carried out in what follows.

(A) Inversion of the equation $t - \beta_1 = L(\phi')$

Since we assume that $A > B > C$, the quantities b', c', and d' , given in equations (19), are all positive. The quantity a' may be either positive or negative, in general. In what follows, we assume that $a' \geq 0$.

We noticed, from equation (8), that $\dfrac{a'}{b'} < \dfrac{c'}{d'}$, (26) since

$$\frac{2C\alpha_1}{h^2} = \frac{AC\omega_{x'}^2 + BC\omega_{y'}^2 + C^2\omega_{z'}^2}{A^2\omega_{x'}^2 + B^2\omega_{y'}^2 + C^2\omega_{z'}^2} ,$$

where $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ are components of the angular velocity w.r.t. the primed systems.

For convenience, the following parameters are defined:

$$n_1^2 \equiv \frac{b'}{a' + b'} , \qquad \text{(a)}$$

$$n_2^2 \equiv \frac{d'}{c' + d'} , \qquad \text{(b)}$$

$$\xi \equiv \frac{1}{\sqrt{(a'+b')(c'+d')}} = \frac{1}{B\sqrt{(A-C)(2A\alpha_1-h^2)}} , \qquad \text{(c)}$$

(27)

$$k \equiv \sqrt{\frac{n_1^2 - n_2^2}{1 - n_2^2}} = \sqrt{\frac{(A-B)(h^2 - 2C\alpha_1)}{(B-C)(2A\alpha_1-h^2)}} , \qquad \text{(d)}$$

$$g \equiv \frac{1}{\sqrt{1 - n_2^2}} = \sqrt{\frac{B(A-C)}{A(B-C)}} , \qquad \text{(e)}$$

$$k' \equiv \sqrt{1 - k^2} = \sqrt{\frac{(A-C)(2B\alpha_1-h^2)}{(B-C)(2A\alpha_1- h^2}} , \qquad \text{(f)}$$

Clearly, $1 \geq n_1^2 > n_2^2 > 0$, and thus $0 < k < 1$, and k' is real since $\frac{a'}{b'} < \frac{c'}{d'}$.

In order to cast expression (a), equation (22) into a more convenient form, we introduce a new variable α by the equation

$$\alpha = \phi' + \pi/2 . \tag{28}$$

It follows immediately, by substituting α and the parameters in (16) into expression (a) of equation (22), that

$$I_2(\phi') = \xi \int_{\alpha_0}^{\alpha} r(\alpha)\, d\alpha , \tag{29}$$

where

$$\alpha_0 = \phi_0' + \frac{\pi}{2} ,$$

$$r(\alpha) = \frac{1}{\sqrt{(1-n_1^2 \sin^2\alpha)(1-n_2^2 \sin^2\alpha)}} . \tag{30}$$

Since the lower limit of integration of equation (29) may be taken to be an absolute constant, we choose $\phi_0' = -\pi/2$ and hence $\alpha_0 = 0$. Therefore, we have

$$I_2(\phi') = \xi \int_0^{\alpha} r(\alpha)\, d\alpha . \tag{31}$$

In what follows we will have occasion to refer to formulas which appear in Byrd and Friedman [1]. Such formula numbers will be indicated by prefixing the numbers with the symbols B-F.

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[1] Byrd, P. and Friedman, M.: Handbook of Elliptic Integrals for Engineers and Physicists. Springer-Verlag, Wurzburg, 1954.

Using B-F (284.00), and expression (a) of (24), we have

$$\int_0^{\alpha} r(\alpha)\, d\alpha = gu = \frac{\quad}{\xi\, AB\sqrt{C}}(t-\beta_1)\ . \tag{32}$$

or, we can write

$$u = \lambda t + \epsilon\ , \tag{33}$$

where

$$\lambda \equiv \frac{\quad}{g\,\xi AB\sqrt{C}} = \sqrt{\frac{(2A\alpha_1-h^2)(B-C)}{ABC}} \qquad \text{(a)}$$

$$\epsilon \equiv -\lambda\beta_1\ . \qquad \text{(b)} \tag{34}$$

Also, from Byrd-Friedman, we have

$$sn^2u \equiv [\ \sin(amu)\ ]^2 = \frac{(1-n_2^2)\ \sin^2\alpha}{1 - n_2^2\ \sin^2\alpha}\ . \tag{35}$$

Solving the above equation for $\sin\alpha$, we write

$$\sin\alpha = \frac{snu}{\sqrt{1 - n_2^2\ cn^2u}}\ , \qquad \text{(a)}$$

and

$$\cos\alpha = \frac{\sqrt{1 - n_2^2}\ cnu}{\sqrt{1 - n_2^2\ cn^2u}}\ , \qquad \text{(b)} \tag{36}$$

where

$$\mathrm{cnu} \doteq \cos(\mathrm{amu}) \; ,$$

and

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1 \; .$$

Since

$$\alpha = \phi' + \frac{\pi}{2} \text{ , we obtain}$$

$$\sin \phi' = - \frac{\sqrt{1-n_2^2}\ \mathrm{cnu}}{\sqrt{1-n_2^2\ \mathrm{cn}^2 u}} \; , \qquad \text{(a)}$$

$$\cos \phi' = \frac{\mathrm{snu}}{\sqrt{1-n_2^2\ \mathrm{cn}^2 u}} \qquad \text{(b)} \qquad (37)$$

$$\tan \phi' = - \frac{\mathrm{cnu}}{g\ \mathrm{snu}} \; . \qquad \text{(c)}$$

The quadrant of ϕ' is uniquely determined by studying the signs of cnu and snu.

Equation (37) is not in a convenient form for calculation, since powers of t appear in the expressions for cnu and snu. This difficulty can be avoided by introducing theta-functions. From B-F(907.01), (907.02), (907.03) (900.04) and (901.01), we have, for $|u| < K'$

$$\mathrm{snu} = u - (1+k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - \qquad \text{(a)}$$
$$- (1 + 135k^2 + 135k^4 + k^6)\frac{u^7}{7!} + \dots,$$

$$\text{cnu} = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!} +$$

$$+ (1 + 408k^2 + 912k^4 + 64k^6)\frac{u^8}{8!} - \dots , \qquad \text{(b)} \qquad (38)$$

$$\text{dnu} = 1 - k^2\frac{u^2}{2!} + (4 + k^2)k^2\frac{u^4}{4!} - (16 + 44k^2 + k^4)k^2\frac{u^6}{6!} + \dots , \qquad \text{(c)}$$

where

$$K' \equiv K(k') , \qquad \text{(a)}$$

$$K \equiv \frac{\pi}{2}\left(1 + 4\sum_{m=1}^{\infty}\frac{q^m}{1+q^{2m}}\right) = \int_0^{\frac{\pi}{2}} \frac{d\nu}{\sqrt{1 - k^2 \sin^2\nu}} , \qquad \text{(b)} \qquad (39)$$

$$q \equiv \frac{1}{2}k_1 \left[1 + 2\left(\frac{k_1}{2}\right)^4 + 15\left(\frac{k_1}{2}\right)^8 + 150\left(\frac{k_1}{2}\right)^{12} + 1707\left(\frac{k_1}{2}\right)^{16} + \dots , \right], \qquad \text{(c)}$$

$$k_1 \equiv \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \quad (k_1^2 < k^2 < 1) \qquad \text{(d)}$$

If we use B-F(1050.01), (1052.01) and (1052.02), we can write

$$\tan\phi' = -\frac{\sqrt{k'}}{g}\,\frac{\cos\nu + q^2\cos 3\nu + q^6\cos 5\nu + \dots}{\sin\nu - q^2\sin 3\nu + q^6\sin 5\nu - \dots} . \qquad (40)$$

where

$$\nu \equiv \frac{\pi}{2K}u .$$

The series (c) of (39) for computing q converges rapidly.

Hence, the angle ϕ' can be expressed in terms of canonical parameters and time through equation (37) and can be computed by using expression (40).

(B) <u>Inversion of the Equation $\phi^* + \beta_2 = M(\phi')$</u>

By using equations (27), (28), (30) and recalling that $\alpha_o = 0$, equation (b) of (24) can be rewritten as

$$\phi^* + \beta_2 = -h\sqrt{C}\,\xi \int_o^{\alpha} [\,(A-B)\cos^2 \quad - A]\, r(\alpha)\, d\alpha\ , \tag{41}$$

From (32), we have

$$\phi^* + \beta_2 = \frac{h}{B}(t-B_1) - h\sqrt{C}\,\xi\,(A-B) \int_o^{\alpha} \cos^2\alpha\ r(\alpha)\, d\alpha\ . \tag{42}$$

Using B-F(284.08) and (432.03), 42 becomes

$$\phi^* + \beta_2 = \frac{h}{B}(t-\beta_1) - \left(\frac{\pi}{2K}\right) [\,\mathcal{A}_5 - u\,\Lambda_o(\beta, k)\,]\ , \tag{43}$$

where

$$\beta \equiv \sin^{-1}\frac{1}{\sqrt{1-\gamma^2}}\ , \qquad \text{(a)}$$

$$\gamma^2 \equiv -\frac{n_2^2}{1-n_2^2}\ . \quad (1 < -\gamma^2 < \infty) \qquad \text{(b)} \tag{44}$$

and the functions $\mathcal{A}_5$ and Λ_o are defined in B-F, Sections 430 and 150, respectively.

Since $u = \lambda(t-\beta_1)$, we can write

$$\phi^* + \beta_2 = M^*(t-\beta_1) - \frac{\pi}{2K}(\mathcal{A}_5 - u)\ , \tag{45}$$

where

$$M^{*} = \frac{h}{B} - \left(\frac{\pi}{2K}\right) [\, 1 - \Delta_{c}(\beta, k)\,]\lambda \quad . \tag{46}$$

(C) Expressions for P_{ϕ}' and θ'

Applying equations (27) and (37), equation (d) of (24) can be written as

$$P_{\phi}' = \frac{\sqrt{C(a'+b')(1-n_2^2)}}{\sqrt{C'}} (k'^2 + k^2\, cn^2 u)^{1/2} \quad . \tag{47}$$

From B-F(121.00), we have

$$dn^2 u = k'^2 + k^2\, cn^2 u \; . \tag{48}$$

Hence, equation (47) takes the form

$$\begin{aligned} P_{\phi}' &= \frac{\sqrt{C(a'+b')(1 - n_2^2)}}{\sqrt{C'}}\, dnu \\ &= \sqrt{\frac{C(2A\alpha_1 - h^2)}{A - C}}\; dnu \; , \end{aligned} \tag{49}$$

and since $P_{\phi}' = h \cos \theta'$, we have

$$\cos \theta' = \frac{P_{\phi}'}{h} = \sqrt{\frac{C(2A\alpha_1 - h^2)}{h^2(A-C)}}\; dnu \; . \tag{50}$$

(D) <u>Inverted solution for the triaxial rigid body problem with no external forces</u>

The general solution for the triaxial rigid body problem with no external forces can then be summarized as follows:

$$\tan \phi' = -\frac{\mathrm{cnu}}{g\,\mathrm{snu}} = -\frac{\sqrt{k'}}{g}\;\frac{\cos \nu + q^2 \cos 3\nu + q^6 \cos 5\nu + \ldots}{\sin \nu - q^2 \sin 3\nu + q^6 \sin 5\nu - \ldots}\,, \tag{a}$$

$$\phi^* + \beta_2 = M^*(t - \beta_1) - \frac{\pi}{2K}(\mathcal{A}_5 - u)\,, \tag{b}$$

$$\psi^* = \beta_3\,, \tag{c}$$

(51)

$$P_{\phi'} = \sqrt{\frac{C(2A\alpha_1 - h^2)}{A - C}}\,\mathrm{dnu}\,, \tag{d}$$

$$P_{\phi^*} = h\,, \tag{e}$$

$$P_{\psi^*} = \alpha_3\,. \tag{f}$$

This solution coupled with equations (12) and (13) gives the complete description of motion of the triaxial body in the space-fixed system $Ox^*y^*z^*$ in terms of the canonical constants and time.

V. <u>Uniaxial Solution</u>

By letting $A = B$, the triaxial solution (51) will reduce to the corresponding uniaxial solution. In order to distinguish between the canonical parameters which appear in triaxial solution and the reduced solution, we will label the latter with the subscript u , that is $(\alpha_{1u}, h_u, \alpha_{3u}, \beta_{1u}, \beta_{2u}, \beta_{3u})$.

For the case $A = B$, we have

$$n_1^2 = n_2^2 = 0\,, \quad k = 0\,, \quad k' = 1\,, \quad g = 1\,,$$

$$\text{snu} = \sin u\,, \qquad \text{cnu} = \cos u$$

$$\lambda = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A-C)}{A^2 C}}$$

Thus from (37) (c), we obtain

$$\phi' - \phi_o = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A-C)}{A^2 C}}\,(t - \beta_{1u})\,. \tag{52}$$

When $A = B$,

$$\beta = \frac{\pi}{2}\,, \quad \Lambda_o(\frac{\pi}{2}, 0) = 1\,, \quad M^* = \frac{hu}{A}\,, \quad \mathcal{Q}_5 = u\,,$$

so that equation (45) reduces to

$$\phi^* = -\beta_{2u} + \frac{h_u}{A}(t - \beta_{1u})\,. \tag{53}$$

Further more, for $A = B$, $dn(u, 0) = 1$, and equation (49) reduces to

$$P_\phi = \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}}\,. \tag{54}$$

Summarizing the uniaxial solution is given as follows:

$$\phi' - \phi'_o = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A-C)}{A^2C}}\,(t - \beta_{1u}) , \quad \text{(a)}$$

$$\phi^* = -\beta_{2u} + \frac{hu}{A}(t - \beta_{1u}) , \quad \text{(b)}$$

$$\psi^* = -\beta_{3u} , \quad \text{(c)}$$

$$P_{\phi'} = \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}} , \quad \text{(d)} \qquad (55)$$

$$P_{\phi^*} = h_u , \quad \text{(e)}$$

$$p_{\psi^*} = \alpha_{3u} , \quad \text{(f)}$$

and the corresponding generator is

$$S_u = -\alpha_{1u}t + h_u\phi^* + \alpha_{3u}\psi^* + \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}}\,(\phi' - \phi'_o) . \qquad (56)$$

Through the use of equations (12) and (13), the complete solution of the force free uniaxial motion can be obtained in the space-fixed system $Ox^*y^*z^*$.

Let us label the parameters which appear in the treatment of the force-free uniaxial problem given in [2] with superscript stars, i.e.,

$(\alpha^*_{1u}, \alpha^*_{2u}, \alpha^*_{3u}, \beta^*_{1u}, \beta^*_{2u}, \beta^*_{3u})$. We have seen that

$$h_u^2 = 2A\,\alpha^*_{1u} + \left(\frac{C-A}{C}\right)\alpha^{*2}_{2u}\,. \tag{57}$$

The corresponding generator, in which we interpreted h_u as a function of α^*_{1u} and α^*_{2u} through (57), takes the form

$$S^*_u = -\alpha^*_{1u}t + \sqrt{2A\,\alpha^*_{1u} + \left(\frac{C-A}{C}\right)\alpha^{*2}_{2u}}\;\phi^* + \alpha^*_{3u}\psi^* + \alpha^*_{2u}(\phi' - \phi'_o). \tag{58}$$

The associated equations of transformation are, after inverting,

$$\phi' - \phi'_o = -\beta^*_{2u} + \frac{\alpha^*_{2u}}{A}\left(\frac{A-C}{C}\right)(t - \beta^*_{1u}), \quad \text{(a)}$$

$$\phi^* = \sqrt{2A\alpha^*_{1u} + \left(\frac{A-C}{C}\right)\alpha^{*2}_{2u}}\;\;(t - \beta^*_{1u}), \quad \text{(b)}$$

$$\psi^* = -\beta^*_{3u}, \quad \text{(c)}$$

$$p_{\phi'} = \alpha^*_{2u}, \quad \text{(d)} \qquad (59)$$

$$p_{\phi^*} = \sqrt{2A\,\alpha^*_{1u} + \left(\frac{C-A}{C}\right)\alpha^*_{2u}}\,, \quad \text{(e)}$$

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[2] Cranford, [illegible]. H. and Fitzpatrick, P. M., "On the influence of Gravity on the Rotational Motion of Artificial Earth Satellites," Environment Induced Orbital Dynamics, Vol. II, Aero-Astrodynamics Laboratory George C. Marshall Space Flight Center, Huntsville, Alabama, October 31, 1967.

$$P_{\psi^*} = \alpha^*_{3u} \ . \qquad \text{(f)}$$

Comparing equations (59) and (55), the parameters $(\alpha^*_{1u}, \alpha^*_{2u}, \alpha^*_{3u}, \beta^*_{1u}, \beta^*_{2u}, \beta^*_{3u})$ and $(\alpha_{1u}, h_u, \alpha_{3u}, \beta_{1u}, \beta_{2u}, \beta_{3u})$ are related as follows:

$$\alpha^*_{1u} = \alpha_{1u} \ , \qquad \text{(a)}$$

$$\alpha^*_{2u} = \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}} \ , \qquad \text{(b)}$$

$$\alpha^*_{3u} = \alpha_{3u} \ , \qquad \text{(c)}$$

$$\beta^*_{1u} = \beta_{1u} + \frac{A}{h_u}\beta_{2u} \ , \qquad \text{(d) (60)}$$

$$\beta^*_{2u} = -\frac{1}{h_u}\sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A-C)}{C}}\ \beta_{2u} \ , \qquad \text{(e)}$$

$$\beta^*_{3u} = \beta_{3u} \ . \qquad \text{(f)}$$

VII. Perturbation of the force free motion of the triaxial rigid body

Recalling Section C, Chapter III, we notice that if we wish to study perturbations of the force free motion of the triaxial rigid body using the canonical perturbation equations of Hamilton-Jacobi theory, it will be necessary to replace θ^* by an equivalent parameter, P_{ψ^*}, the momentum conjugate to ψ^*. Thateither θ^* or P_{ψ^*} will give an equivalent description of the motion follows from equation (b) of (13). It is clear from this equation that the momentum conjugate to any angle ψ which lies in the x^*y^*-plane is independent of the angle ψ and only depends on h and θ^*. Therefore

$$P_\psi = P_{\psi^*} = h \cos \theta^* = P_{\phi^*} \cos \theta^* , \tag{61}$$

and thus the six independent quantities (ϕ^*, ϕ', ψ^*, P_{ϕ^*}, $P_{\phi'}$, P_{ψ^*}) will describe the motion of the triaxial rigid body with respect to the $OX^*Y^*Z^*$ system completely. The Hamilton function from which ϕ^*, ϕ', P_{ϕ^*}, and $P_{\phi'}$ are to be obtained is, of course, still given by (10). Furthermore, we can consider H as given by (10) to be the Hamilton function of an extended system of variables (ϕ^*, ϕ', ψ^*, p_{ϕ^*}, $p_{\phi'}$, p_{ψ^*}), which satisfy the canonical equations o fmotion,

$$\dot{\phi}^* = \frac{\partial H}{\partial p_{\phi^*}} , \quad (a) \qquad \dot{p}_{\phi^*} = - \frac{\partial H}{\partial \phi^*} ; \quad (d)$$

$$\dot{\phi}' = \frac{\partial H}{\partial p_{\phi'}} , \quad (b) \qquad \dot{p}_{\phi'} = - \frac{\partial H}{\partial \phi'} , \quad (e) \qquad (62)$$

$$\dot{\psi}^* = \frac{\partial H}{\partial p_{\psi^*}} , \quad (c) \qquad \dot{p}_{*} = - \frac{\partial H}{\partial \psi^*} , \quad (f)$$

subject to the constraints

$$\psi^* = \text{constant} , \quad \text{(a)}$$

$$p_{\psi^*} = h \cos \theta^* = \text{constant}. \quad \text{(b)} \qquad (63)$$

This follows from the fact that the two differential equations (c) and (f) of (62), which have been added to the system, are entirely consistent with the equations of constraint (63)

VIII. Gravity-gradient Potential for Triaxial Body

The gravity-gradient potential V for the triaxial body is given by

$$V = -\frac{3}{2}\kappa \ [(A-C) \cos^2 \chi + (A-B) \cos^2 \beta] , \qquad (64)$$

where $K = n'^2$ and n' is the mean motion of the Earth about the triaxial body. We consider a circular orbit, for which K is a constant. The angles α, β, and χ are the direction angles of the line segment from the c.m. of the body to the c.m. of the Earth with respect to the axes $ox'y'z'$, the principal axes of the body. Since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, equation (64) can be rewritten as

$$V = \frac{3}{2}\kappa \ (B-A) + W \qquad (65)$$

where

$$W = \left(\frac{B-C}{A-C}\right) W_1 + W_2 , \qquad (66)$$

and

$$W_1 = -\frac{3}{2}\kappa\,(B-C)\cos^2\chi\,,$$
$$W_2 = \frac{3}{2}\kappa\,(A-B)\cos^2\alpha\,. \tag{67}$$

The expression for $\cos\chi$ in terms of canonical constants and time t has been previously deduced [1] and can be written in the form

$$\cos\chi = D_1\cos\theta' + D_2\sin\theta'\sin\phi^* + D_3\sin\theta'\cos\phi^*\,, \tag{68}$$

where

$$D_1 = \sin i\sin\ell\cos\theta^* - [\cos i\sin\ell\cos(\Omega+\beta_3) + \cos\ell\sin(\Omega+\beta_3)]\sin\theta^*\,,$$
$$D_2 = -\cos i\sin\ell\sin(\Omega+\beta_3) + \cos\ell\cos(\Omega+\beta_3)] \tag{69}$$
$$D_3 = -\sin i\sin\ell\sin\theta^* - [\cos i\sin\ell\cos(\Omega+\beta_3) + \cos\ell\sin(\Omega+\beta_3)]\cos\theta^*.$$

Note that D_1, D_2 and D_3 are functions of three canonical constants only, viz., $\alpha_2 = h$, α_3 and β_3, and contain t explicitly only through ℓ and Ω, which are both linear in t .

A suitable expression for $\cos\alpha$ can be derived similarly. From spherical trigonometry, we have

$$\cos\alpha = \cos\phi'\cos\phi_H - \sin\phi'\sin\phi_H\cos\theta' \tag{70}$$

and

$$\cos\theta_H = \cos i\cos\theta^* + \sin i\sin\theta^*\cos(\Omega+\beta_3). \tag{71}$$

Introducing

$$E_1 \equiv \cos(\phi^* - \phi_H) = \frac{\cos i - \cos\theta_H \cos\theta^*}{\sin\theta_H \sin\theta^*} \tag{72}$$

$$E_2 \equiv \sin(\phi^* - \phi_H) = -\frac{\sin i \sin(\Omega + \beta_3)}{\sin\theta_H} \tag{73}$$

We can write equation (7) in the form

$$\cos\alpha = E_1 (\cos\phi' \cos\phi^* - \cos\theta' \sin\phi' \sin\phi^*) - E_2(\cos\phi' \sin\phi^* + \cos\theta' \sin\phi' \cos\phi^*). \tag{74}$$

Note that E_1 and E_2 are functions of only three canonical constants, viz., α_2, α_3 and β_3 and contain t explicitly only through ℓ and Ω. It is important to note that D_1, D_2, D_3, E_1 and E_2 do not contain the moments of inertia A, B, and C. Thus these coefficients can be treated as constants when we expand $\cos\chi$ and $\cos\alpha$ in Taylor's series about their values at $B = A$. The reason for the expansion is that the angles ϕ^*, ϕ' and θ' for the unperturbed triaxial body are no longer either constant or simple linear functions of the time, as was the case in the uniaxial problem. Thus, anticipating difficulties in the integration of the perturbation equations, we try to linearize the arguments of the trigonometric functions which will appear in the integration.

Introducing the notation

$$f(\chi) \equiv \cos\chi, \quad g(\alpha) \equiv \cos\alpha \tag{75}$$

We treat $f(\chi)$ and $g(\alpha)$ as functions of B and expand about the value $B = A$. Using prime notation to indicate derivatives with respect to B, we have

$$f(\chi) = f(B) - f'(B)\,(A-B) + \frac{1}{2} f''(B)\,(A-B)^2 + O[(A-B)^3] \tag{76}$$

where

$$f(B) = D_1[\cos\theta']_{B=A} + D_2[\sin\theta' \sin\phi^*]_{B=A} + D_3[\sin\theta' \cos\phi^*]_{B=A}$$

$$\begin{aligned} f'(B) &= D_1[\frac{\partial}{\partial B}\cos\theta']_{B=A} + D_2[\frac{\partial}{\partial B}(\sin\theta' \sin\phi^*)]_{B=A} \\ &+ D_3[\frac{\partial}{\partial B}(\sin\theta' \cos\phi^*)]_{B=A} \end{aligned} \tag{77}$$

$$\begin{aligned} f''(B) &= D_1[\frac{\partial^2}{\partial B^2}\cos\theta']_{B=A} + D_2[\frac{\partial^2}{\partial B^2}(\sin\theta' \sin\phi^*)]_{B=A} \\ &+ D_3[\frac{\partial^2}{\partial B^2}(\sin\theta' \cos\phi^*)]_{B=A} \end{aligned}$$

and

$$g(\alpha) = g(B) - g'(B)\,(A-B) + O[(A-B)^2] \tag{78}$$

where

$$\begin{aligned} g(B) &= E_1[\cos\phi' \cos\phi^* - \cos\theta' \sin\phi' \sin\phi^*]_{B=A} \\ &+ E_2[\cos\phi' \sin\phi^* + \cos\theta' \sin\phi' \cos\phi^*]_{B=A} \end{aligned} \tag{79}$$

$$\begin{aligned} g'(B) &= E_2[\frac{\partial}{\partial B}(\cos\phi' \cos\phi^* - \cos\phi' \sin\phi' \sin\phi^*)]_{B=A} \\ &+ E_2[\frac{\partial}{\partial B}(\cos\phi' \sin\phi^* + \cos\theta' \sin\phi' \cos\phi^*)]_{B=A} \end{aligned}$$

In equation (78), only two terms have been carried since $g(\alpha)$ is multiplied by the fact r (A-B) in W.

From equations (66), (67), (76) and (78), we obtain

$$W = \left(\frac{B - C}{A - C}\right) W_{1u} + W_{2t} + O[(A-B)^3] , \qquad (80)$$

where

$$W_{1u} = - \frac{3\kappa}{2}(A-C)\ [f(B)]^2 .$$

$$W_{2t} = \frac{3\kappa}{2}(A-B) \left\{ 2(B-C)\ f(B)\ f'(B) + [g(B)]^2 \right\}$$
$$- \frac{3\kappa}{2}(A-B)^2 \left\{ (B-C)\ [f'(B)]^2 + f(B) . f''(B)] + 2g(B)\ g'(B) \right\} .$$

These expressions for W_{1u} and W_{2t} can be used to study the perturbations of the variables ($\alpha 1$, h , $\alpha 3$, β_1 , β_2 , β_3) which are given by the following relations

$$\dot{\alpha}_i = \left(\frac{B-C}{A-C}\right) \frac{\partial W_{1u}}{\partial \beta i} + \frac{\partial W_{2t}}{\partial \beta i} ,$$

$$\dot{\beta} i = - \left(\frac{B-C}{A-C}\right) \frac{\partial W_{1u}}{\partial \alpha_i} - \frac{\partial W_{2t}}{\partial \alpha_i} \quad (i = 1,2,3), (\alpha_2 = h) \qquad (81)$$